Part 2

Levered exchange-traded products: theory and practice[1]

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Abstract
The introduction of levered exchange-traded products was heralded as a convenient mechanism for investors to enhance performance over traditional borrowing and leverage strategies. In many cases, these products have not generated the anticipated benefits. Because multi-period returns for levered and inverse products can depend on the path of the underlying asset's returns, the rebalancing strategy is a crucial determinant of their success. The standard ETF approach is to rebalance on an end-of-day daily basis. Naive investors may base their expectations of these products on the expected performance of traditional "buy and hold" leverage. Optimal rebalancing decisions depend upon several interrelated factors, including the expected return pattern of the underlying asset and the investor's time horizon. Empirical tests illustrate the pros and cons of two types of levered products under various scenarios. We find that in a majority of outcomes, term borrowing performs better than end-of-day daily rebalance leverage and increasingly so as volatility increases and holding periods expand. Daily rebalance leverage performs better in trending and in certain extreme market conditions.

1 The authors would like to thank Hongseok Namkoong for his efforts in running the simulation model. The opinions and viewpoints expressed are those of the authors and are for informational purposes only, and should not be construed as a recommendation of any specific security or strategy. Investors should always consult an investment professional before making any investment. One of the authors has a business relationship with EdgeShares LLC.

Introduction

The traditional approach for levering a portfolio utilizes a margin loan from a brokerage firm to invest the proceeds in the target securities. This approach can be expensive for individual investors. As an alternative, exchange-traded products (ETPs) and related securities have been developed to offer investors leverage without direct borrowing by the investor. These products are often used in accounts where traditional leverage is restricted (e.g., self-directed retirement accounts). These products are called levered and inverse products in this report.

ETPs can be less expensive than traditional leverage due to economies of scale and other features. For instance, the originating firm can employ futures contracts, swap agreements, and other derivative instruments to increase efficiency. Accordingly, leverage ETPs have grown in the amount of assets under management. Since their introduction in the U.S. in 2006, levered and inverse ETFs have grown to over U.S.$30.2 billion (leveraged to U.S.$13.5 billion and inverse to U.S.$16.7 billion) in assets by year-end 2012 – about 2.2% of the overall U.S. ETF marketplace.[2]

This growth has not come without controversy. For investors with horizons longer than one day, the standard levered ETF products have not provided returns in concert with their anticipated performance.[3] Two of the more egregious examples are cited by the Financial Industry Regulatory Authority (FINRA) in regulatory notice #09-31 (June 2009) regarding Nontraditional ETFs. For "most leveraged and inverse ETFs," FINRA states that "due to the effect of compounding, performance over longer periods of time can differ significantly from the anticipated performance (or inverse of performance) of their underlying index during the same period of time. For example, between December 1, 2008 and April 30, 2009:

- The Dow Jones U.S. Oil & Gas Index gained 2%, while an ETF seeking to deliver twice the index's daily return fell 6% and the related ETF seeking to deliver twice the inverse of in the index's daily return fell 26% (Figure 1).

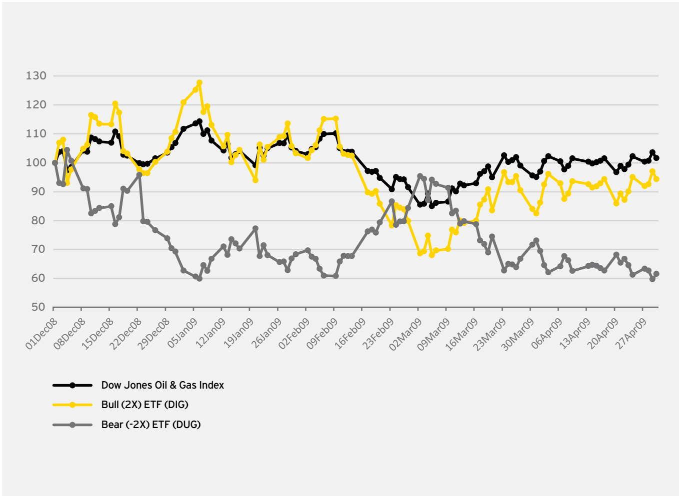

Figure 1: Dow Jones Oil & Gas Index and related Bull (2X) and Bear (-2X) ETFs
1 December 2008 (= 100.0) to 30 April 2009

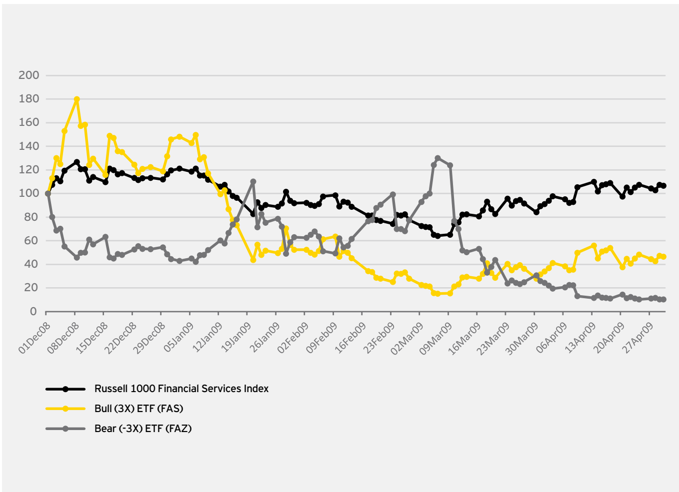

Figure 2: Russell 1000 Financial Services Index and related Bull (3X) and Bear (-3X) ETFs
1 December 2008 (= 100.0) to 30 April 2009

2 Source: IndexUniverse.com, Dave Nadig, and Olly Ludig, "ETF fund flows: GDX Adds $370.6M," 1 January 2013.
3 Naive investors often fail to understand the adverse consequences of daily rebalancing inherent in many of these products.

- An ETF seeking to deliver three times the daily return of the Russell 1000 Financial Services Index fell 53% while the index actually gained around 8%. The related ETF seeking to deliver three times the inverse of the index's daily return declined by 90% over the same period (Figure 2)."[4]

Over this five-month period, the Dow Jones U.S. Oil & Gas Index experienced 48% annualized volatility of returns, while the Russell 1000 Financial Services Index suffered 91% annualized volatility. Periods of high volatility are particularly difficult for most levered and inverse products, for reasons shown in this report.

At a fundamental level, a levered exchange-traded product (ETP) is a combination — portfolio – of an investment in the underlying security ("underlying" herein) plus an amount of borrowed cash. Rebalancing decisions determine the mix of these two "assets" at selected time points over the life of the product. For the most part, difficulties with levered products spring from the type and frequency of rebalancing the level of equity and borrowing. For example, consider an initial $100 investment in a double-levered fund. At the start, we can view the levered investment as two parts: $200 in equity, and $100 in borrowing. Next, assume that equity increases by 10% to $220 during the first trading day. At this point, the security is no longer a doubled-levered fund; it is a 2.2 times levered fund while the investors initial stake has grown to $120. Typically, to maintain the two times leverage ratio, the portfolio manager will rebalance the fund before the start of the next trading day, called daily rebalancing, herein. Borrowing an additional $20 will further enhance the fund's equity position, currently $220. The $20 can be employed to purchase an incremental $20 stake in the underlying security, so that the mix is now $240 equity and $120 borrowing — again doubled levered.[5] Assume that the equity decreases by 9.09% over the next trading day so that the underlying asset returns to its initial price. Now the fund's $240 in equity declines to $218.18 while the amount borrowed remains at $120. At this point the fund is a 1.82 times levered fund. To restore the fund to 2.0 times leverage, $21.82 in equity must be sold and the proceeds returned to the lender. Once this happens, the fund will have $196.36 in equity with $98.18 in borrowing to yield a leverage ratio of 2.0. In this case, although the underlying equity's value has remained the same over the 2 day period (+10%, –9.09%), the investor's initial stake of $100 has now shrunk to $98.18 for a 2-day loss of 1.82%. Had the fund not rebalanced at the end of the first trading day, the investor's stake would have returned to $100 at the end of the second trading day leaving the investor flat for the two days.

This simple example shows how the performance of a levered product depends upon the return patterns of the underlying security **and** the rebalancing decisions. As shown in this paper, the latter issue becomes especially pertinent during periods of high volatility and major market moves.

Theory of rebalancing

This section analyzes the performance of levered products based on previous research on rebalancing the assets in a portfolio over time. Rebalancing can be done in a variety of ways. The simplest way is to not rebalance at all — called the "do-nothing" or "buy-and-hold" strategy. This approach is the one assumed by the traditional Markowitz portfolio model over the investment planning horizon (for example one year ahead). We call this approach "point-to-point" or "term borrowing" leverage.

We can separate rebalancing decisions into two fundamental types: 1) momentum based, and 2) fixed/target proportions.[6] The distinguishing feature between the two involves the decision to purchase or sell the underlying asset during a period of increasing or decreasing performance. For momentum based rebalancing, we add to the amount of equity as equities increase — as discussed in the previous section; whereas in fixed/target rebalancing, we decrease the equity level. The situation reverses when the equity returns are negative, i.e., selling equity for momentum based, and purchasing equity in target proportions. Buying equities during increasing return periods (and selling during market downturns) fits within the context of "portfolio-insurance" strategies. In contrast, selling equities during bull

4 Non-traditional ETFs: FINRA reminds firms of sales practice obligations relating to leveraged and inverse exchange-traded funds," FINRA Regulatory Notice 09-31, https://www.finra.org/web/groups/industry/@ip/@reg/@notice/documents/notices/p118952.pdf, June 2009.

5 An alternative process would be to return part of the capital to the investor as would be the case if the investor were utilizing futures contracts, but this step is rarely, if ever, done with levered ETPs.

6 The fixed/target proportion-based rebalancing strategy should not be confused with the end-of-day daily rebalancing typically employed by constant leverage strategies. In the former, outperforming assets are sold, whereas with the latter, they are bought. Likewise, with the former strategy, underperforming assets are bought, whereas with the latter, they are sold.

markets fits the fixed-mix strategy. Perold and Sharpe (1988) and Tokat (2007) discuss the pros and cons of these opposing strategies.

To start, we propose a stochastic process for the underlying equity element within a self-contained fund. The standard approach models equity prices with geometric Brownian motion (GBM). First, we compare the buy-and-hold (do nothing) portfolio with a strategy that constantly rebalances from the perspective of a portfolio of n-assets. Recall the performance of the buy-and-hold strategy from the Markowitz model. Suppose there are n securities whose mean return is μ, and covariance matrix Σ. Assuming normality, the average buy-and-hold portfolio return with weights, w^{BH}, does not have a closed form expression as the sum of the log normal random variables is not a log normal random variable. However, assuming that the number of the securities in the portfolio is large enough, one can approximate the return as a normal random variable. That is, $r^{BH} \sim N(w^T\mu, \sigma_p^2) \equiv N(w^T\mu, w^T\Sigma w)$.

Next, consider the rebalanced portfolio constructed from the same securities with the same weight () as the previous buy-and-hold portfolio. Since it is rebalanced at every intermediate juncture, security prices must be modeled as stochastic processes. We model them as an n-dimensional geometric Brownian motion whose return distribution for a unit time length is the same as in the previous case. Then, the price process of security can be written as the following SDE: $dS_t^i/S_t^i = (\mu_i + \sigma_i^2/2)dt + dB_t^i$, where σ_i^2 is the i-th diagonal term of Σ (hence, variance of stock i) and for the Cholesky factorization of Σ, L and the standard n-dimensional Wiener process $(W_t^1 \cdots W_t^n)^T, d(B_t^i \cdots B_t^n)^T = Ld(W_t^1 \cdots W_t^n)^T$.

Since the portfolio is rebalanced continuously to the initial weight (w), its instantaneous growth rate is the same as the weighted sum of instantaneous growth rates of the constituent securities at any given juncture. Consequently, the SDE for the portfolio wealth can be written as:

$$\frac{dP_t^{FM}}{P_t^{FM}} = \sum_{i=1}^{n} w_i \frac{dS_t^i}{S_t^i} = \sum_{i=1}^{n} w_i \left\{ \left(\mu_i + \frac{\sigma_i^2}{2} \right) dt + dB_t^i \right\}.$$

With a little algebra, one can show that, for the standard 1-dimensional Wiener process W_t,

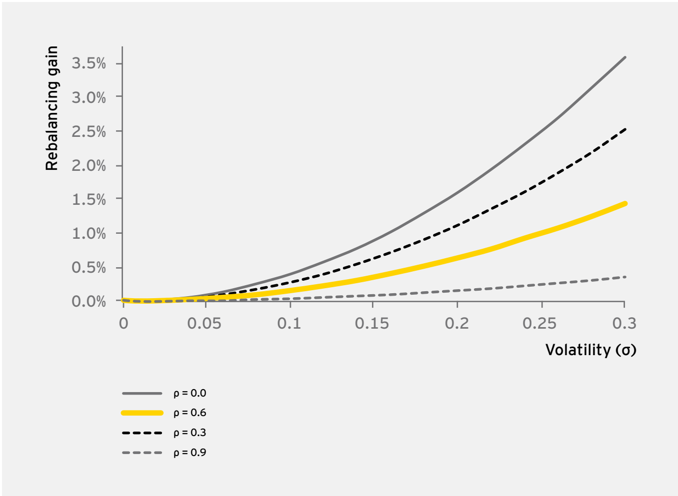

Figure 3: Effects of volatility and correlation on rebalancing gains

$$\frac{dP_t^{FM}}{P_t^{FM}} = \left(w^T\mu + \frac{1}{2}\sum_{i=1}^{n} w_i\sigma_i^2 \right) dt + \sigma_p^2 dW_t.$$

Hence, the return of the fixed mix portfolio for a unit time length can be given as

$$r^{FM} \sim N\left(w^T\mu + \frac{1}{2}\sum_{i=1}^{n} w_i\sigma_i^2 - \frac{1}{2}\sigma_p^2, \sigma_p^2 \right) \equiv N\left(w^T\mu + \frac{1}{2}\sum_{i=1}^{n} w_i\sigma_i^2 - \frac{1}{2}w^T\Sigma w, w^T\Sigma w \right)$$

Consequently, returns of both buy-and-hold and continuously rebalanced fixed mix are normally distributed with the same variance σ_p^2, while the mean of the latter one contains extra terms, $(\Sigma_{i=1}^{n} w_i\sigma_i^2 - \sigma_p^2)/2$. These extra terms, which are referred to as rebalancing gain or volatility pumping [Luenberger (1998), Mulvey and Kim (2008)], represent the value of an option to rebalance the portfolio to the target fixed-proportions.

To observe its effects more closely, consider the following example: suppose we have n securities where the expected return and the volatility of each are μ_0 and σ, and the correlation is ρ. Assuming that the portfolio is equally weighted, the amount of the rebalancing gain (RG) is:

$$RG = \frac{1}{2}\left\{ \sum_{i=1}^{n} \frac{1}{n}\sigma^2 - \left(\frac{1}{n} \cdots \frac{1}{n} \right) \Sigma \left(\frac{1}{n} \cdots \frac{1}{n} \right)^T \right\} = \frac{(n-1)\sigma^2(1-\rho)}{2n}.$$

Now it is evident that the continuously rebalanced strategy has benefit over the static buy-and-hold rule, even without mean-reversion: the rebalancing gain is always positive, except in the case where all security returns are perfectly correlated, in which it becomes zero. Note that the rebalancing gain is an increasing function of the number of securities n and the volatility σ and is a decreasing function of the correlation ρ. Therefore, with the wisdom from the portfolio theory, one can see that volatile securities should not be penalized when a portfolio is constructed with the fixed mix rule, as long as their correlations to other securities are low and they have reasonable and positive expected returns. They can serve as good sources of rebalancing gains, while their risks can be effectively eliminated via diversification. See Figure 3 for the graphical illustrations for the effects of to rebalancing gains from Monte-Carlo simulations.

In general, rebalancing gains accrue when the better performing assets are systematically sold and the underperforming assets are systematically bought — as shown above. The critical feature is to lower risks when risky assets do well relative to other assets, such as with the fixed-mix investment strategy. Of course, rebalancing gains can be turned into rebalancing losses when the expected returns (the drift terms) are negative or when the better performing assets are systematically bought and the underperforming assets are systematically sold (as is often the case when a leveraged portfolio is rebalanced to a pre-set leverage ratio on a discrete time basis, such as daily).

Rebalancing decisions are critical to performance, along with the standard performance paths. Rebalancing gains are generally difficult to achieve with levered and inverse funds on their own, outside a generic portfolio. Daily rebalancing (DR) leverage strategies increase equities and beta risks during upward movements, rather than during down moves as is the case with the discussion above. Thus, in trendless and volatile markets, the drift term for DR strategies is negative because they are buying more of the outperforming assets rather than selling them [i.e., the constant leverage trap (CLT)].[7]

In addition, transaction costs and market impact costs affect performance in the context of rebalancing a portfolio. The general approach for addressing transaction costs for fixed mix investors is to construct a no-trade zone around the target allocation percentage and to not trade while asset levels remain within the designated zone. For example, suppose that we are interested in preserving a 50/50 fund. Then, we might select a zone equal to 49% to 51%. In this situation, the fund would only rebalance the portfolio to the 50% target when prices fell or gained enough to place it outside the no-trade zone. In this example, trading is done on a price driven basis, as compared with time dependent trading rules such as daily rebalancing [see Mulvey and Simsek (2002), and Kritzman and Page (2009) for further details]. The advantages of fixed mix strategies are best expressed in markets that are trendless with relatively high volatility and low transaction costs.

Empirical tests

In this section, we explore the boundaries of the buy and hold [term borrowing (TB) leverage] and daily rebalance leverage strategies and give some insight into when one is superior to the other in a controlled set of experiments.

In general, there is a tradeoff between the probability of positions experiencing early termination with TB strategies and the re-levering gains and losses associated with DR strategies. With TB strategies, there is no re-levering; the main concern is being cashed out prematurely on an adverse move. In practice, TB strategies will likely self-liquidate before losing all their capital.[8] However, TB strategies generally do not suffer volatility-induced asset erosion. Thus, if the TB strategy does not terminate early, then one can predict with relative certainty how much the TB strategy will be worth for a given underlying asset value at the end of a target term.[9] Conversely, daily rebalance strategies are path-dependent. For multi-periods, one cannot predict with any certainty how much a DR strategy will be worth at the end of the term. As volatility and noise increase, the end-of-day DR-levered strategies become increasingly vulnerable to rebalancing losses because they increase borrowing and exposure only after the underlying asset appreciates and decrease borrowing and exposure only after the underlying asset depreciates. In essence, they generally buy high and sell low, subjecting them to the sometimes severe negative consequences of noise in markets (i.e., "volatility" losses).

7 Remember that the stated Brownian motion processes must have positive drift terms.

8 An early termination of a TB fund would require the investor to re-establish his position in another security if he intended to maintain his position.

9 The possibility of early termination for TB strategies makes them partially path-dependent.

Many investors find a daily DR strategy counterintuitive from the standpoint of temporal considerations, and prefer a TB strategy with a fixed term when the investor's horizon is longer than one day. With a TB strategy, the amount borrowed stays constant while the leverage ratio floats. With an end-of-day DR strategy, the amount borrowed floats so that the leverage ratio remains constant on a daily basis (Table 1). We note that over 90% of the 274 levered and inverse-levered ETFs available in the marketplace as of September 30, 2012 employ an end-of-day DR strategy.[10] TB solutions for the ETF marketplace are extremely limited; an exception is EdgeShares LLC,[11,12] which has developed an effective solution for achieving targeted leverage in a TB strategy.

Through our simulated return environments, we outline the boundaries of when TB strategies outperform DR strategies, and vice versa. For each of the two types of strategies (TB and DR), we simulate four distinct leveraged and inverse-leveraged strategies: double leverage (2X), triple leverage (3X), double inverse leverage (-2X) and triple inverse leverage (-3X).[13]

In our experiments, we evaluate 21 expected period returns (μ) ranging from –50% to +50% in increments of 5%, along with 18 annualized volatilities (σ) ranging from 5% to 90%, also in increments of 5% for the underlying risky asset. We simulate two time periods: 126 trading days, representing a half-year period, and 252 trading days, representing a full trading year. We group all the simulation trials together (21 x 18 x 250,000) and sort outcomes based on the realized return (r) and realized

Strategy	Code	Amount borrowed	Rebalance frequency	Leverage ratio	Other names
Term borrowing	TB	Fixed amount for term	None	Floats, never reset	Point-to-point leverage
Daily rebalance	DR	Updates daily	Daily	Rebalanced to target at end of day	Constant proportional leverage

Table 1: Characteristics of two rebalancing strategies

Index name	Time frame	Return	Volatility	2X ETF return	-2X ETF return
Dow Jones Oil & Gas Index	1 Dec 2008 to 30 April 2009	2%	45%	-6.0%	-26.0%

Table 2A: Dow Jones Oil & Gas Index and ETFs (1 Dec 2008– 30 April 2009)

standard deviation (σ) of each simulation.[14] We then select only those outcomes that result in realized returns that are within ±0.5% of a target return (r_i) and realized volatilities that are also within ±0.5% of a target volatility (σ_j). All other simulation results are discarded for the tables. Furthermore, because all (r_i, σ_j) are not equally likely, results for some (r_i, σ_j) pairs that are deemed incredibly unlikely (e.g., r_i = - 50%, σ_j = 5%) are not shown and appear as blank cells in the tables. For all simulated underlying risky asset return series pertaining to a respective outcome pair (r_i, σ_j), the returns of each of the four variations of simulated TB and DR strategies are then generated by overlaying each respective strategy (eight in all) onto each daily return series of the underlying risky asset. The results for each strategy variation are then averaged into each respective outcome pair (r_i, σ_j) for that respective variation.

We employ a stylized framework to compare strategies by assuming a virtual perfect world for leveraged and inverse-leveraged investors. We assume no borrowing costs, transaction

10 See BlackRock ETP 2012 Landscape Global Handbook at http://www.indexfunds.com.cn/userfiles/file/1358232962976.pdf

11 See Kiron, K., "Securitization system and process" United States Patent Application 20110191234. Filed February 2, 2011.

12 See Kiron, K., "Securitization system and process II" United States Patent Application 20130046673. Filed August 15, 2012.

13 The 2X strategy simulated borrowing an amount equal to an investor's initial equity and investing so that the investor received 200% of the return of the underlying risky asset. The 3X strategy simulated borrowing an amount equal to twice the investor's initial equity and investing the entire amount so that the investor received 300% of the return of the underlying risky asset. The –2X strategy simulated shorting an amount equal to twice the investor's initial equity in the risky asset so that the investor received –200% of the return of the underlying risky asset. Likewise, the –3X strategy simulated shorting an amount equal to three times the investor's initial equity in the risky asset so that the investor received –300% of the underlying return of the risky asset.

14 We sort and analyze the Monte Carlo simulation results based on realized return and realized volatility to directly compare the likely return differences between (a) term borrowing (buy and hold) leveraged and inverse-leveraged strategies with those of (b) constant daily rebalanced leverage and inverse-leverage strategies for given market outcomes (return and volatility combinations). This allows comparison of the average returns of each strategy when the paths taken by the underlying asset to generate a given return and volatility combination vary from run to run.

costs, management fees, and taxes.[15] We did not model initial and maintenance margin amounts and assumed no credit or counterparty risk.[16] In short, we focused on the interaction between time and volatility to differentiate the two strategies. Each simulation of the underlying asset price changes is modeled as a geometric Brownian motion (GBM) continuous time pricing process using the multiplicative form to discretize the price process. The result is the following formula:

$$S(t_{k+1}) = e^{\gamma\Delta t+\sigma\epsilon(t_k)\sqrt{\Delta t}}S(t_k)$$

Where, μ = expected period return = –50%, –45%, ... , +50%
σ= target volatility = 5%, 10%, ... , 90%
$\gamma = \mu - \frac{1}{2}\sigma^2$
t = time
S(t) = asset price at time t_k
n = number of days in period
k = 0 ... n-1

A standard variance reduction method – antithetic variables – is employed to reduce the standard errors over all runs. Thus, for each pair of simulations, the sign of the random terms for each sequence is opposite that of its corresponding paired simulation. This increases the accuracy of the results by reducing the variance of the simulated paths. Thus, to run 250,000 trials for each given pair, we generate only 125,000 unique sequences of random variables.

Results of the experiments

Sorting by realized return and variance allows us to identify the relative advantages of TB versus end-of-day DR strategies.[17] As a test of the model's robustness, we refer to the two examples cited by FINRA at the beginning of this paper. In the first case, the index gained 2%, while the 2X ETF fell 6% and the related –2X ETF fell 26%. The annualized volatility of the Dow Jones U.S. Oil & Gas Index during the 1 Dec 2008 to 30 April 2009 period was 45% (Table 2A). Over a simulated six-month period, when realized volatility is 45% and return of the underlying is 0%, a 2X DR strategy returns an estimated –9.6% while it returns an estimated –0.3% when the return of the underlying is 5% (Table 2B). When realized volatility is 45% and return of the underlying is 0% over six months, a –2X DR strategy returns an estimated –26.2% while it returns an estimated –33.1% when the return of the underlying is 5%. Although, the time period is slightly longer, these results are roughly in-line with this example.

In the second example, the 3X ETF fell 53% while the index actually gained around 8%. The related – 3X ETF declined by 90% over the same period. The annualized volatility of the Russell 1000 Financial Services Index during the same five-month period was 91% (Table 3A). Over six months when realized volatility is

2X simulated six-month horizon							
Asset return	Annualized volatility	Naïve E(return)	2X TB E(return)	Prob(TB termination)	2X DR E(return)	DR 1-way turnover	TB – DR E(return)
0%	45%	0%	0%	--	-9.6%	556%	9.6%
5%	45%	10%	10.0%	--	-0.3%	583%	10.3%
-2X simulated six-month horizon							
0%	45%	0%	-2.7%	2.7%	-26.2%	506%	23.5%
5%	45%	-10%	-13.7%	4.1%	-33.1%	484%	19.4%

Table 2B: Simulated leverage and inverse DB and TB strategies
*TB – term borrowing; DR – daily rebalance; E(return) – expected return

Index name	Time frame	Return	Volatility	3X ETF return	-3X ETF return
Russell 1000 Financial Services Index	1 Dec 2008–30 April 2009	8%	91%	-53.0%	-90.0%

Table 3A: Russell 1000 Financial Services Index and ETFs (1 Dec 2008–30 April 2009)

3X simulated six-month horizon							
Asset return	Annualized volatility	Naïve E(return)	3X TB E(return)	Prob(TB termination)	3X DR E(return)	DR 1-way turnover	TB – DR E(return)
5%	90%	15%	-24.8%	34.6%	-66.5%	2,843%	41.6%
10%	90%	30%	-10.0%	30.7%	-61.4%	3,061%	51.4%
-3X simulated six-month horizon							
5%	90%	-15%	-70.2%	64.9%	-93.4%	1,658%	23.2%
10%	90%	-30%	-78.8%	69.8%	-94.3%	1,552%	15.5%

Table 3B: Simulated leverage and inverse leverage strategies
*TB – term borrowing; DR – daily rebalance; E(return) – expected return

15 For the sake of brevity, we excluded borrowing and transaction costs. Since transaction costs will vary depending on market and implementation, turnover data is available from the authors. Borrowing costs are not a major differentiating factor between the two strategies particularly in a low-interest rate environment. Initially, borrowing costs will be the same for both strategies.
16 In practice, no regulatory authority or counterparty would knowingly let an investor's equity totally evaporate and leverage ratio sky rocket before taking action.
17 Because of our simplifying assumptions, real-world outcomes may differ considerably from what we present here.

Target underlying asset realized semiannual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50				-24.3	-24.1	-23.8	-23.4	-22.9	-22.5	-21.9	-21.3	-20.7	-20.1	-19.4	-18.7	-18.0	-17.2	-16.5
-45				-19.6	-19.3	-19.2	-19.2	-19.2	-19.2	-19.3	-19.2	-19.1	-18.9	-18.5	-18.0	-17.6	-16.9	-16.4
-40			-15.5	-15.2	-14.8	-14.4	-13.9	-13.7	-13.6	-13.7	-13.7	-13.9	-14.0	-14.1	-13.9	-13.8	-13.3	-12.8
-35			-11.7	-11.3	-10.9	-10.3	-9.7	-9.1	-8.7	-8.4	-8.3	-8.6	-8.5	-8.7	-8.5	-8.4	-8.8	-8.6
-30			-8.4	-8.0	-7.4	-6.8	-6.0	-5.2	-4.5	-3.7	-3.4	-3.1	-2.9	-3.3	-3.1	-3.2	-3.0	-3.5
-25		-5.9	-5.6	-5.1	-4.5	-3.7	-2.9	-1.9	-0.9	0.0	0.8	1.6	1.9	2.1	1.9	2.1	2.1	2.5
-20		-3.7	-3.3	-2.7	-2.0	-1.2	-0.2	0.9	2.1	3.3	4.6	5.4	6.1	6.7	7.5	7.6	7.6	7.2
-15	-2.1	-1.9	-1.4	-0.8	0.0	0.9	2.0	3.3	4.7	6.1	7.6	8.9	9.9	11.0	11.3	12.0	12.5	12.4
-10	-0.9	-0.6	-0.1	0.6	1.5	2.6	3.8	5.2	6.8	8.4	10.1	11.8	13.1	14.4	15.6	16.7	16.8	18.0
-5	-0.1	0.2	0.8	1.5	2.5	3.7	5.1	6.7	8.4	10.3	12.2	14.1	16.1	17.9	19.5	20.0	21.0	21.8
0	0.1	0.5	1.1	2.0	3.1	4.4	5.9	7.6	9.6	11.7	13.9	16.2	18.2	20.4	22.2	23.7	25.3	25.9
5	-0.1	0.3	1.0	1.9	3.1	4.6	6.3	8.2	10.3	12.6	15.1	17.7	20.0	22.5	24.7	26.8	28.4	29.3
10	-0.8	-0.4	0.3	1.4	2.7	4.3	6.1	8.2	10.6	13.1	15.9	18.7	21.6	24.3	26.9	29.4	31.7	33.0
15	-2.1	-1.6	-0.8	0.4	1.8	3.5	5.6	7.8	10.4	13.2	16.2	19.2	22.5	25.6	28.7	31.2	34.0	36.2
20		-3.3	-2.4	-1.1	0.4	2.3	4.5	7.0	9.8	12.8	16.1	19.5	23.1	26.5	29.9	33.2	36.3	38.3
25		-5.4	-4.5	-3.1	-1.4	0.6	3.0	5.7	8.7	12.0	15.5	19.2	23.1	26.9	30.8	34.4	37.7	40.7
30			-7.0	-5.6	-3.8	-1.6	1.0	3.9	7.2	10.7	14.6	18.6	22.8	27.0	31.3	35.1	39.4	41.7
35			-10.1	-8.6	-6.6	-4.2	-1.4	1.7	5.2	9.0	13.2	17.5	22.1	26.8	31.2	35.7	39.8	43.3
40			-13.7	-12.0	-9.9	-7.3	-4.4	-1.0	2.8	6.9	11.4	16.1	20.9	25.9	30.9	36.4	40.8	44.7
45				-15.9	-13.7	-10.9	-7.7	-4.1	-0.1	4.3	9.1	14.1	19.4	24.9	30.4	35.9	41.1	45.5
50				-20.3	-17.9	-15.0	-11.6	-7.7	-3.4	1.3	6.4	11.9	17.5	23.4	29.4	35.6	40.8	47.1

Table 4A: 2X term borrowing (TB) leverage minus 2X daily rebalance (DR) leverage
Average TB – DR realized semiannual return differences (%)

90% and return of the underlying is 5% and 10%, respectively, the 3X DR strategy returns an estimated –66.5% and –61.4%, respectively (Table 3B). For the same terms: the –3X DR strategy returns an estimated –93.4% and –94.3%, respectively. Again, although the time period is slightly longer, our simulated results are similar to this real-world example.[18]

It is worth noting that in both examples illustrated above, for similar return and risk characteristics, on average TB strategies would have handily outperformed the DBL strategies (Tables 2B and 3B). However, for the index and time period in question, a 3X TB strategy would have experienced early termination when the actual index was down by 34.8% from 1 Dec 2008 to 6 March 2009.

Tables 4A through 7A show the semiannual return differences between TB and DR strategies for the four leverage and inverse leverage levels modeled (2X, –2X, 3X, –3X), while Tables 4B through 7B show the annual results. Overall, in very-low-volatility directional markets, DR strategies generally outperform TB strategies because there are minimal reversals while the amount levered (relative to the initial stake) continually increases as the underlying asset generally moves in one's favor, thereby magnifying gains. Similarly, if the underlying asset generally moves in an adverse direction, there are minimal reversals while leverage is steadily decreased, thereby mitigating losses and preventing the threat of early termination in all but the most extreme of circumstances.

There is a broad range of return outcomes where the 2X TB strategies dominate the 2X DR strategies (see Tables 4A and 4B). In particular, TB overwhelmingly dominates when underlying asset returns are modest (-5% to +5%) and increasingly (see light shaded area in tables) so as volatility and holding period increase.

Overall, 2X TB generally outperforms 2X DR in an ever-expanding realized return space plume spreading from lower to higher volatility, and increasingly so for longer holding periods (annual versus semiannual). The only caveat to this generalization is when the underlying asset moves in an extreme adverse direction, then the back leg of the plume tapers off as the model's measured probability of early termination for TB strategies starts to become a significant factor. Tables 5A and 5B show similar results for the –2X strategies.

18 In this case, with volatility at these extreme levels, subjecting a DR strategy to an extra month of volatility losses compared to the results from the actual ETF can account for some of the observed difference. Other factors may be at work as well. For example, the simulations assume the investor is able to execute at the closing level and have no market impact. During crash periods, certain funds may not even be able to trade, let alone trade at or near the close.

Target underlying asset realized annual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50			-24.3	-23.9	-23.4	-22.8	-22.0	-21.2	-20.3	-19.4	-18.4	-17.3	-16.3	-15.2	-14.1	-13.0	-12.0	-11.0
-45			-19.6	-19.3	-19.3	-19.5	-19.6	-19.6	-19.3	-18.8	-18.0	-17.3	-16.4	-15.5	-14.5	-13.3	-12.4	-11.3
-40		-15.6	-15.2	-14.6	-14.1	-13.8	-13.9	-14.1	-14.4	-14.3	-14.4	-13.7	-13.7	-13.0	-12.0	-11.5	-10.3	-9.5
-35		-11.8	-11.3	-10.6	-9.7	-8.9	-8.5	-8.7	-8.8	-9.0	-9.1	-9.0	-9.2	-8.4	-8.3	-8.2	-7.4	-6.1
-30		-8.5	-7.9	-7.1	-6.0	-4.9	-4.1	-3.6	-3.5	-3.6	-3.6	-3.5	-3.8	-3.9	-4.2	-4.0	-2.6	-1.5
-25		-5.7	-5.0	-4.0	-2.8	-1.5	-0.1	0.9	1.5	1.5	2.3	2.0	1.3	1.2	1.3	1.6	2.0	3.0
-20	-3.8	-3.4	-2.6	-1.5	-0.1	1.5	3.1	4.7	5.6	6.3	6.8	6.7	6.8	6.7	6.7	6.5	7.3	7.5
-15	-2.1	-1.5	-0.6	0.6	2.1	4.0	6.0	7.9	9.3	10.8	11.3	12.0	11.8	11.4	11.9	11.9	11.8	11.7
-10	-0.8	-0.2	0.8	2.2	3.9	6.0	8.3	10.6	12.7	14.2	15.8	16.5	16.5	17.0	16.6	16.5	17.3	16.6
-5	0.0	0.6	1.8	3.3	5.2	7.5	10.1	12.9	15.4	17.8	19.2	20.2	20.9	21.7	22.3	22.0	22.0	23.3
0	0.2	1.0	2.2	3.9	6.0	8.6	11.5	14.6	17.8	20.3	22.6	24.4	25.3	26.9	26.5	27.5	28.2	27.1
5	0.0	0.8	2.2	4.1	6.4	9.2	12.4	15.9	19.5	22.8	25.5	27.5	29.4	30.5	32.5	31.8	33.3	34.7
10	-0.7	0.2	1.7	3.7	6.3	9.4	12.9	16.7	20.8	24.5	27.9	31.0	32.7	34.5	36.1	36.8	37.8	37.7
15	-1.9	-0.9	0.7	2.9	5.7	9.1	12.9	17.2	21.6	26.2	29.9	33.7	36.0	38.5	39.4	40.5	41.8	44.3
20	-3.6	-2.6	-0.8	1.6	4.7	8.4	12.5	17.2	22.2	27.1	31.6	35.7	38.8	41.7	42.8	45.2	47.3	48.3
25		-4.7	-2.8	-0.1	3.2	7.2	11.7	16.8	22.2	27.7	33.2	37.3	41.2	45.5	46.5	51.1	52.2	53.2
30		-7.3	-5.2	-2.4	1.2	5.5	10.4	15.9	21.7	27.8	33.3	39.0	44.0	47.2	51.3	53.6	55.1	55.3
35		-10.4	-8.2	-5.1	-1.2	3.4	8.7	14.6	20.9	27.7	33.9	39.5	45.4	49.8	53.6	56.8	59.4	60.6
40		-14.0	-11.6	-8.3	-4.1	0.8	6.5	12.9	19.7	27.0	33.8	40.5	46.5	52.3	56.8	60.1	62.1	66.5
45			-15.5	-12.0	-7.5	-2.2	3.9	10.7	18.2	25.9	33.3	40.4	47.7	53.5	59.4	63.1	67.5	69.5
50			-19.9	-16.1	-11.3	-5.6	0.9	8.2	16.1	24.5	32.7	40.8	48.0	53.7	61.0	66.7	69.7	74.0

Table 4B: 2X term borrowing (TB) leverage minus 2X daily rebalance (DR) leverage

Average TB – DR realized annual return differences (%)

Target underlying asset realized semiannual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50				-173.2	-161.2	-146.9	-131.0	-113.3	-94.4	-74.6	-54.5	-33.8	-14.3	4.4	21.5	37.9	52.9	66.6
-45				-119.3	-109.3	-97.5	-84.1	-69.5	-53.7	-37.3	-20.6	-4.5	11.2	26.8	40.0	50.3	64.1	72.3
-40			-87.2	-80.4	-71.9	-62.0	-50.7	-38.4	-25.1	-11.4	2.5	15.6	28.5	40.4	50.1	60.4	68.6	74.6
-35			-58.0	-52.2	-45.0	-36.5	-26.8	-16.2	-4.9	6.7	17.9	29.0	39.3	48.0	56.7	64.0	70.4	73.6
-30			-36.9	-31.9	-25.6	-18.2	-9.9	-0.7	9.0	18.9	28.8	37.4	45.1	51.9	58.8	63.7	66.8	73.0
-25		-24.8	-21.7	-17.3	-11.8	-5.3	1.9	9.9	18.3	26.6	34.6	42.0	47.4	52.7	57.4	59.8	64.0	64.9
-20		-13.8	-11.0	-7.1	-2.3	3.4	9.8	16.8	23.9	31.1	37.2	42.3	47.4	51.3	53.2	55.9	58.9	60.4
-15	-7.8	-6.3	-3.8	-0.4	3.9	9.0	14.7	20.7	26.8	32.4	37.3	41.2	45.1	46.9	49.3	50.8	52.6	54.5
-10	-3.0	-1.6	0.6	3.7	7.5	12.0	17.1	22.4	27.4	32.2	34.8	38.7	41.3	42.5	44.4	44.8	46.2	45.0
-5	-0.4	0.8	2.8	5.6	9.0	13.1	17.6	22.1	26.1	29.4	32.2	34.6	35.4	36.9	37.2	39.4	38.7	39.8
0	0.4	1.5	3.3	5.8	8.9	12.5	16.5	20.1	23.5	25.4	27.5	28.3	29.2	29.9	31.4	31.5	32.2	33.3
5	-0.4	0.6	2.3	4.6	7.4	10.7	14.1	16.8	19.4	20.8	21.5	22.0	22.5	22.7	24.6	23.6	24.9	25.3
10	-2.3	-1.4	0.1	2.2	4.7	7.6	10.3	12.4	14.1	14.8	15.0	14.8	15.3	16.9	16.6	16.5	17.9	18.0
15	-5.3	-4.5	-3.1	-1.2	1.1	3.6	5.6	7.1	7.6	7.6	8.1	8.0	8.4	9.6	9.4	10.7	10.7	10.9
20		-8.4	-7.1	-5.4	-3.3	-1.3	0.1	0.8	0.6	0.4	0.7	1.0	1.2	2.0	2.8	3.5	4.5	6.0
25		-13.0	-11.8	-10.2	-8.5	-7.2	-6.5	-6.7	-6.6	-6.8	-6.3	-6.0	-5.5	-4.4	-3.0	-2.1	-0.7	0.8
30			-17.1	-15.7	-14.4	-13.9	-13.7	-13.9	-14.2	-13.7	-13.2	-12.4	-11.7	-10.1	-8.9	-7.1	-5.2	-4.0
35			-22.9	-21.8	-21.2	-21.1	-21.4	-21.7	-21.2	-20.4	-19.2	-18.1	-16.7	-14.8	-13.1	-11.2	-9.5	-7.4
40			-29.2	-28.8	-28.8	-29.0	-28.7	-28.2	-27.1	-25.6	-24.0	-22.5	-20.4	-18.1	-16.1	-14.0	-12.3	-10.3
45				-36.4	-36.3	-35.8	-34.7	-33.2	-31.4	-29.3	-27.3	-25.0	-22.7	-20.4	-18.1	-16.0	-14.0	-12.0
50				-41.5	-40.2	-38.5	-36.7	-34.6	-32.5	-30.2	-27.8	-25.5	-23.1	-20.8	-18.5	-16.4	-14.4	-12.5

Table 5A: -2X term borrowing (TB) leverage minus -2X daily rebalance (DR) leverage

Average TB – DR realized semiannual return differences (%)

Target underlying asset realized annual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50			-172.2	-153.4	-130.7	-104.8	-76.8	-48.0	-19.5	6.5	30.2	50.2	65.9	78.6	91.2	97.1	101.3	103.8
-45			-118.0	-102.4	-83.6	-62.2	-39.0	-15.5	7.7	27.5	45.4	60.3	72.6	82.5	91.9	92.4	98.8	98.2
-40		-88.9	-79.1	-66.0	-50.1	-32.0	-12.6	6.7	25.5	42.2	55.5	65.9	76.0	82.6	84.2	90.0	90.2	90.8
-35		-59.3	-50.9	-39.8	-26.2	-10.8	5.3	21.6	36.3	49.3	58.8	67.5	73.7	76.5	81.1	82.6	83.5	81.5
-30		-37.8	-30.6	-20.9	-9.2	4.1	17.9	31.1	43.1	51.7	60.2	65.6	69.1	73.1	73.9	75.9	73.1	74.5
-25		-22.4	-16.1	-7.7	2.6	13.9	25.6	36.7	45.3	52.5	58.2	62.4	66.2	66.0	66.3	66.9	67.3	64.5
-20	-15.0	-11.6	-6.0	1.4	10.4	20.4	30.3	38.7	45.2	50.7	55.4	56.9	60.2	58.9	58.5	59.6	58.6	56.8
-15	-7.3	-4.3	0.6	7.2	15.1	23.7	31.8	38.1	43.1	47.4	48.7	51.1	50.5	52.2	50.8	50.9	51.4	51.2
-10	-2.5	0.2	4.6	10.5	17.4	24.8	31.5	35.9	39.3	41.5	43.8	44.2	43.8	43.3	43.5	44.4	44.5	42.6
-5	0.0	2.5	6.4	11.7	17.9	24.2	28.5	31.8	34.4	36.0	36.2	37.3	35.6	37.0	37.5	37.4	36.7	35.3
0	0.7	2.9	6.5	11.3	16.7	21.5	24.9	26.9	28.1	29.7	29.0	29.3	29.3	28.8	29.9	29.0	29.2	30.8
5	0.0	2.0	5.2	9.5	14.1	17.7	20.0	21.4	21.3	21.9	21.9	22.1	22.0	23.5	22.8	23.6	23.1	21.0
10	-2.0	-0.2	2.7	6.6	10.3	12.8	14.0	14.2	14.7	15.2	15.0	14.5	16.5	15.7	17.4	17.9	17.0	18.0
15	-5.0	-3.4	-0.7	2.7	5.6	6.4	6.7	7.0	6.9	7.9	7.5	9.1	9.6	10.6	12.1	12.2	12.0	12.3
20	-8.9	-7.4	-4.9	-2.1	-0.3	0.1	-0.3	-0.4	0.4	0.8	1.2	3.5	4.4	5.9	6.6	7.3	7.6	9.1
25		-12.1	-9.9	-7.7	-7.0	-7.3	-7.6	-7.3	-6.8	-5.4	-4.3	-2.3	-1.1	0.5	2.3	2.9	4.4	5.2
30		-17.4	-15.4	-14.2	-14.3	-14.7	-14.6	-14.1	-12.5	-11.7	-8.8	-6.9	-5.0	-3.2	-1.6	-0.1	1.1	2.4
35		-23.2	-21.7	-21.5	-22.0	-22.0	-21.4	-20.0	-17.8	-15.7	-13.0	-10.5	-8.3	-6.2	-4.3	-2.8	-1.2	0.1
40		-29.5	-28.9	-29.3	-29.3	-28.2	-26.7	-24.3	-21.8	-18.7	-15.9	-13.1	-10.7	-8.4	-6.4	-4.5	-3.0	-2.0
45			-36.7	-36.3	-34.9	-32.8	-30.0	-27.0	-23.8	-20.6	-17.5	-14.6	-11.9	-9.6	-7.5	-5.7	-4.3	-3.1
50			-41.3	-39.2	-36.7	-33.7	-30.6	-27.3	-24.0	-20.7	-17.6	-14.8	-12.2	-9.8	-7.8	-6.1	-4.7	-3.6

Table 5B: –2X term borrowing (TB) leverage minus –2X daily rebalance (DR) leverage

Average TB – DR realized annual return differences (%)

Target underlying asset realized semiannual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50				-11.6	-11.2	-10.8	-10.2	-9.7	-9.0	-8.4	-7.7	-7.1	-6.4	-5.8	-5.1	-4.5	-4.0	-3.4
-45				-15.5	-15.0	-14.4	-13.7	-12.9	-12.1	-11.2	-10.3	-9.5	-8.6	-7.7	-6.9	-6.1	-5.3	-4.6
-40			-20.7	-20.2	-19.5	-18.7	-17.8	-16.8	-15.7	-14.6	-13.5	-12.3	-11.2	-10.1	-9.0	-7.9	-6.9	-6.0
-35			-26.4	-25.7	-24.9	-23.8	-22.7	-21.4	-20.1	-18.7	-17.2	-15.7	-14.3	-12.8	-11.5	-10.1	-8.9	-7.7
-30			-23.2	-23.2	-23.3	-23.1	-22.7	-21.9	-20.9	-19.8	-18.3	-16.9	-15.4	-13.7	-12.3	-10.8	-9.4	-8.2
-25		-16.5	-15.7	-14.8	-14.2	-14.0	-14.2	-14.3	-14.0	-13.6	-13.2	-12.4	-11.4	-9.8	-8.7	-7.7	-6.3	-4.7
-20		-10.4	-9.4	-8.2	-6.8	-5.8	-5.2	-5.0	-5.0	-5.4	-5.4	-4.9	-4.1	-3.5	-2.2	-1.9	-0.3	0.1
-15	-6.1	-5.5	-4.3	-2.8	-1.0	1.0	2.3	3.1	3.4	3.4	3.2	3.9	3.6	4.4	4.6	5.6	6.2	6.9
-10	-2.6	-1.8	-0.5	1.4	3.6	6.2	8.4	10.1	11.5	11.9	12.1	12.3	12.4	12.5	13.7	14.1	14.4	15.3
-5	-0.4	0.5	2.1	4.2	6.9	10.0	13.2	15.8	18.1	19.7	20.8	20.5	20.8	21.4	22.6	22.3	23.2	24.0
0	0.4	1.5	3.3	5.8	8.9	12.5	16.5	20.1	23.4	25.7	27.4	28.7	29.4	29.5	30.5	31.2	32.1	31.9
5	-0.3	0.9	3.0	5.9	9.5	13.8	18.5	23.1	27.3	30.5	33.2	35.7	37.1	38.1	38.5	40.6	40.9	41.6
10	-2.6	-1.1	1.3	4.6	8.7	13.6	19.0	24.6	30.2	34.7	38.7	42.0	45.1	45.4	48.5	47.7	50.1	51.4
15	-6.5	-4.8	-2.0	1.7	6.4	12.0	18.3	24.7	31.5	37.0	42.5	46.9	50.0	52.4	54.7	56.2	58.6	60.1
20		-10.1	-7.0	-2.8	2.6	8.9	16.0	23.6	31.5	38.5	45.1	51.0	55.5	58.4	61.9	63.1	66.3	67.3
25		-17.2	-13.7	-8.9	-2.9	4.2	12.3	21.0	29.9	38.8	46.8	53.3	58.7	63.9	67.5	69.9	73.1	75.5
30			-22.2	-16.8	-10.0	-2.0	7.0	16.9	27.1	37.0	46.5	54.5	61.8	67.9	73.5	76.9	82.0	83.8
35			-32.5	-26.5	-18.9	-10.0	0.1	11.1	22.8	34.2	45.2	55.1	64.5	71.6	77.2	83.2	88.1	89.0
40			-44.8	-38.0	-29.6	-19.7	-8.4	3.9	16.7	30.3	42.8	54.8	65.0	74.0	80.6	87.9	93.0	96.0
45				-51.5	-42.2	-31.2	-18.7	-5.0	9.2	24.2	38.5	52.2	65.0	74.6	84.1	91.8	99.2	103.7
50				-67.0	-56.7	-44.5	-30.8	-15.6	0.4	17.0	33.1	49.1	61.5	74.2	85.2	98.1	103.1	113.1

Table 6A: 3X term borrowing (TB) leverage minus 3X daily rebalance (DR) leverage

Average TB – DR realized semiannual return differences (%)

Target underlying asset realized annual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50			-11.6	-11.0	-10.3	-9.5	-8.6	-7.6	-6.7	-5.8	-4.9	-4.1	-3.4	-2.7	-2.2	-1.7	-1.3	-1.0
-45			-15.5	-14.7	-13.7	-12.6	-11.4	-10.2	-8.9	-7.7	-6.6	-5.5	-4.5	-3.7	-2.9	-2.3	-1.8	-1.3
-40		-20.9	-20.1	-19.1	-17.8	-16.4	-14.9	-13.2	-11.6	-10.1	-8.6	-7.2	-5.9	-4.8	-3.8	-3.0	-2.3	-1.7
-35		-26.6	-25.6	-24.3	-22.7	-20.9	-18.9	-16.9	-14.8	-12.8	-10.9	-9.1	-7.5	-6.1	-4.9	-3.8	-2.9	-2.2
-30		-23.4	-23.3	-23.5	-23.0	-21.9	-20.3	-18.3	-16.1	-14.0	-11.9	-9.8	-8.0	-6.3	-5.0	-3.7	-2.7	-1.9
-25		-15.9	-14.7	-14.3	-14.6	-14.7	-14.2	-13.8	-12.2	-10.7	-8.7	-7.1	-5.4	-3.9	-2.6	-1.5	-0.4	0.3
-20	-10.8	-9.7	-7.9	-6.1	-5.5	-5.7	-6.2	-5.5	-5.1	-4.5	-3.1	-2.1	-1.3	0.6	2.0	2.7	3.3	4.4
-15	-5.9	-4.6	-2.4	0.3	2.2	2.8	2.9	3.0	3.0	3.0	3.9	5.3	5.7	6.5	7.9	8.2	9.1	8.7
-10	-2.3	-0.7	1.9	5.2	8.5	10.5	11.3	11.6	11.3	11.9	11.6	12.2	12.7	13.6	14.6	15.2	15.1	14.4
-5	-0.1	1.8	4.8	8.9	13.3	16.6	18.6	19.3	20.2	19.8	20.7	20.4	21.4	22.2	22.1	21.8	21.7	22.3
0	0.7	2.9	6.5	11.3	16.7	21.6	25.3	26.8	28.6	29.4	29.3	29.9	30.1	29.9	30.6	29.9	29.1	27.3
5	0.1	2.6	6.8	12.3	18.8	25.1	29.9	33.4	35.6	36.4	36.8	39.1	38.2	38.4	39.2	38.2	37.6	38.9
10	-2.1	0.8	5.6	11.9	19.5	27.1	33.6	38.9	42.8	43.8	46.4	47.5	47.6	48.5	47.9	45.4	45.0	45.5
15	-5.9	-2.6	2.8	10.0	18.7	28.1	36.8	43.4	47.3	52.1	53.7	54.9	55.8	56.8	55.8	54.8	55.5	54.5
20	-11.5	-7.7	-1.5	6.7	16.6	27.5	37.6	46.6	52.3	57.8	61.8	63.2	64.4	65.1	65.0	64.3	62.4	61.5
25		-14.5	-7.6	1.7	13.0	25.3	37.4	48.5	57.6	63.2	68.0	70.5	74.3	73.2	74.5	76.8	72.5	71.8
30		-23.1	-15.3	-4.9	7.8	21.8	36.3	49.3	58.8	67.9	72.9	78.0	81.3	83.0	83.5	83.4	83.5	78.2
35		-33.6	-24.9	-13.2	0.9	16.9	33.0	48.7	61.2	71.6	78.8	84.9	89.3	90.0	90.3	94.2	91.8	88.8
40		-46.0	-36.3	-23.3	-7.5	10.2	28.7	46.3	61.6	73.4	83.9	91.1	95.7	100.6	100.1	98.3	99.9	99.0
45			-49.7	-35.3	-17.8	2.1	22.7	43.2	60.8	75.2	85.9	94.7	102.0	106.5	109.3	110.9	109.7	109.1
50			-65.1	-49.1	-29.8	-7.9	15.4	38.2	59.8	76.5	91.0	100.2	107.8	113.4	114.7	118.9	120.0	124.0

Table 6B: 3X term borrowing (TB) leverage minus 3X daily rebalance (DR) leverage

Average TB – DR realized annual return differences (%)

Target underlying asset realized semiannual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50				-447.3	-403.5	-353.2	-299.1	-242.1	-184.9	-129.1	-77.3	-28.6	13.7	48.5	78.2	102.0	121.8	134.9
-45				-291.6	-258.3	-220.3	-179.0	-136.2	-93.0	-52.4	-13.3	21.1	50.8	74.7	94.3	109.0	123.3	128.9
-40			-208.4	-187.2	-161.3	-131.9	-99.9	-66.7	-33.9	-2.4	25.3	50.5	70.0	87.7	101.2	112.2	116.8	120.3
-35			-133.1	-116.2	-95.8	-72.4	-47.2	-21.2	4.2	27.0	48.4	65.7	80.0	92.0	100.4	105.3	109.5	112.2
-30			-81.4	-67.8	-51.3	-32.5	-12.3	7.9	27.6	45.5	59.7	72.8	84.2	87.8	94.5	98.3	98.7	102.8
-25		-54.3	-46.0	-35.0	-21.4	-6.1	10.2	26.7	41.0	53.3	63.4	74.4	77.7	82.4	87.1	87.9	88.8	86.6
-20		-29.2	-22.4	-13.2	-2.0	10.5	23.6	36.0	46.8	55.5	63.8	68.4	72.2	74.6	75.3	77.2	77.5	76.0
-15	-16.4	-12.9	-7.2	0.5	9.9	20.3	30.5	39.8	46.8	53.0	58.5	60.6	63.7	64.6	65.6	66.2	65.3	65.4
-10	-6.1	-3.1	1.7	8.2	16.1	24.6	32.6	39.5	44.3	48.5	49.5	52.7	53.8	54.1	55.7	54.6	54.5	52.5
-5	-0.8	1.8	5.9	11.5	18.1	24.8	30.7	34.4	38.3	39.7	41.1	42.6	42.8	42.8	43.1	45.2	43.3	41.7
0	0.7	2.9	6.5	11.2	16.7	21.7	25.7	28.0	29.8	30.5	31.0	31.6	31.7	32.9	33.3	33.1	32.4	33.3
5	-0.7	1.2	4.2	8.3	12.6	16.1	17.8	19.3	20.0	20.2	20.7	20.7	22.1	22.3	23.9	22.6	24.0	23.2
10	-4.5	-2.9	-0.2	3.2	6.3	7.9	8.4	9.2	9.8	9.8	10.5	11.0	11.8	13.8	13.9	14.7	14.9	15.5
15	-10.2	-8.8	-6.4	-3.8	-2.3	-1.7	-1.6	-1.5	-1.2	-0.9	1.4	2.1	4.3	5.4	6.0	7.3	8.2	8.3
20		-16.1	-14.1	-12.7	-12.7	-12.7	-12.4	-11.5	-10.5	-9.1	-6.8	-5.0	-3.1	-1.0	0.3	1.4	3.0	3.7
25		-24.6	-23.3	-23.3	-23.4	-23.1	-22.0	-20.2	-18.0	-15.4	-12.9	-10.4	-7.8	-6.0	-4.0	-2.4	-1.0	0.2
30			-34.4	-34.0	-32.7	-30.7	-28.0	-25.2	-22.1	-19.1	-16.1	-13.3	-10.8	-8.5	-6.6	-4.8	-3.5	-2.3
35			-37.8	-35.8	-33.5	-30.8	-27.8	-24.8	-21.7	-18.7	-15.9	-13.2	-10.8	-8.7	-6.8	-5.3	-4.0	-3.0
40			-33.9	-32.1	-30.0	-27.5	-24.9	-22.2	-19.4	-16.7	-14.2	-11.8	-9.7	-7.8	-6.1	-4.7	-3.6	-2.7
45				-28.8	-26.9	-24.7	-22.4	-19.9	-17.5	-15.0	-12.7	-10.6	-8.7	-7.0	-5.5	-4.2	-3.2	-2.4
50				-26.0	-24.3	-22.3	-20.2	-18.0	-15.7	-13.5	-11.5	-9.5	-7.8	-6.3	-4.9	-3.8	-2.9	-2.1

Table 7A: –3X term borrowing (TB) leverage minus –3X daily rebalance (DR) leverage

Average TB – DR realized semiannual return differences (%)

Target underlying asset realized annual return ±0.5(%)	Target underlying asset realized annualized volatility ±0.5(%)																	
	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
-50			-443.0	-375.0	-297.4	-215.5	-135.0	-61.2	1.5	49.2	88.1	116.5	130.9	135.2	140.6	138.3	133.5	131.7
-45			-287.0	-235.8	-177.1	-115.7	-55.2	-3.1	41.6	74.6	99.8	119.1	124.9	129.1	130.6	123.9	121.8	114.8
-40		-213.8	-182.9	-143.2	-97.9	-50.6	-5.6	33.9	65.2	87.9	105.4	112.7	119.1	118.5	114.4	112.8	106.5	101.9
-35		-136.7	-112.4	-81.1	-45.3	-9.3	24.4	52.2	75.0	90.1	100.7	105.7	107.5	103.9	103.1	98.3	94.3	86.5
-30		-84.0	-64.4	-39.3	-10.9	17.7	43.6	62.5	77.6	88.1	93.2	94.3	94.5	93.3	90.3	87.7	79.5	74.6
-25		-48.0	-32.0	-11.5	11.3	33.0	51.1	66.2	75.3	81.6	84.5	85.9	84.9	80.7	77.7	73.5	68.3	63.8
-20	-32.2	-23.8	-10.6	6.2	24.5	41.6	54.1	63.0	69.2	72.2	73.5	73.9	72.9	69.8	64.6	60.8	56.2	53.5
-15	-15.3	-8.3	2.7	16.7	31.3	43.5	51.3	57.8	61.6	62.4	62.5	61.2	60.1	57.8	53.3	50.4	48.3	45.2
-10	-5.1	0.8	10.1	21.7	32.8	41.2	46.7	49.7	51.3	51.1	52.8	51.7	48.5	46.0	43.7	40.8	38.6	37.7
-5	0.1	5.1	13.0	22.5	30.7	35.5	37.8	40.2	40.9	42.8	40.0	39.7	37.4	36.1	34.2	32.7	31.4	29.6
0	1.5	5.8	12.5	20.1	25.4	27.9	28.7	29.8	29.6	30.8	30.7	29.6	28.1	27.4	26.4	24.6	24.0	22.3
5	-0.1	3.6	9.4	14.8	17.5	18.5	19.1	20.2	20.6	20.5	20.9	21.6	20.4	20.5	18.4	18.7	16.9	15.0
10	-4.0	-0.7	4.0	7.4	7.9	8.3	8.6	8.8	10.6	12.5	13.0	12.3	13.6	13.0	13.1	12.2	11.7	11.4
15	-9.7	-6.9	-3.4	-2.3	-2.5	-2.7	-1.6	0.1	2.5	4.4	5.4	7.1	7.7	8.2	8.2	7.7	7.2	6.8
20	-17.0	-14.5	-12.7	-13.2	-13.2	-12.2	-10.0	-7.5	-4.3	-2.0	-0.2	1.8	3.0	3.6	3.7	4.0	3.8	3.7
25		-23.6	-23.8	-23.9	-22.5	-19.8	-16.5	-12.9	-9.5	-6.4	-4.1	-2.0	-0.8	0.2	1.0	1.2	1.3	1.6
30		-34.7	-34.2	-31.9	-28.3	-24.2	-19.9	-15.7	-12.0	-8.8	-6.1	-4.1	-2.6	-1.5	-0.7	-0.3	0.1	0.3
35		-38.2	-35.4	-31.9	-27.8	-23.5	-19.3	-15.3	-11.8	-8.7	-6.3	-4.4	-2.9	-1.9	-1.2	-0.7	-0.4	-0.2
40		-34.2	-31.7	-28.5	-24.9	-21.0	-17.2	-13.7	-10.5	-7.8	-5.6	-3.9	-2.6	-1.7	-1.1	-0.6	-0.4	-0.2
45			-28.5	-25.7	-22.4	-18.9	-15.5	-12.3	-9.4	-7.0	-5.1	-3.5	-2.3	-1.5	-0.9	-0.6	-0.3	-0.2
50			-25.8	-23.2	-20.2	-17.1	-14.0	-11.1	-8.5	-6.3	-4.5	-3.2	-2.1	-1.4	-0.8	-0.5	-0.3	-0.2

Table 7B: –3X term borrowing (TB) leverage minus –3X daily rebalance (DR) leverage
Average TB – DR realized annual return differences (%)

3X leveraged strategy results are shown in Tables 6A and 6B while –3X results are found in Tables 7A and 7B. In general, TB strategies are generally superior at all modest return levels and increasingly so as volatility and holding period increase. It is worthwhile noting that when the underlying asset moves in one's favor accompanied by high volatility, for the range of outcomes we examine, one is usually better off with TB leverage.

Our results confirm what other researchers have noted [Bush (2009), and Cheng and Madhavan (2009)]. In almost every scenario and time period studied, the TB strategy does increasingly better than end-of-day DR strategy as volatility increases. For example, for a six-month holding period, when the underlying asset returns 10%, the 2X TB strategy outperforms the 2X DR strategy by an average of 0.3% when realized volatility is 15%, by an average of 4.3% at 30% volatility, by an average of 10.6% at 45% volatility, and by an average of 18.7% when realized volatility is 60% (See Table 8).

For readers interested in delving deeper into these topics, the raw expected returns of the four respective TB and DR strategy simulations (2X, –2X, 3X and –3X), the model's measured probability of early termination for each TB strategy outcome and the expected average one-way turnover associated with each DR strategy outcome are available upon request.

Conclusions

This paper has shown that the best strategy for rebalancing a levered fund depends upon, among other things, the expected pattern of returns of the underlying target security and the investor's time horizon. There are two fundamental types of rebalancing strategies: 1) momentum-based, such as portfolio insurance, and 2) fixed-mix (also called fixed-proportions). The main difference involves the change in the amount of the risky assets, such as equity, during market increases and decreases. Momentum strategies increase exposure to risky assets during sustained prices rises, whereas the risky assets are sold during price rises for fixed-mix strategies. For levered funds, the traditional end-of-day daily rebalancing to target leverage approach is a momentum strategy. The term-based borrowing strategy can be interpreted as a buy-and-hold approach.

If one is to rebalance a portfolio on a regular basis in the context of fixed-mix strategies, there are distinct advantages

			2X six month horizon			
Asset return	Annualized volatility	2X TB E(return)	Prob(TB termination)	2X DR E(return)	DR 1-wayturnover	TB – DR E(return)
10%	15%	20%	--	19.7%	207%	0.3%
10%	30%	20%	--	15.7%	411%	4.3%
10%	45%	20%	--	9.4%	609%	10.6%
10%	60%	19.8%	0.1%	1.2%	809%	18.7%
			2X one year horizon			
10%	15%	20%	--	18.3%	415%	1.7%
10%	30%	20%	--	10.6%	820%	9.4%
10%	45%	19.6%	0.3%	-1.1%	1206%	20.8%
10%	60%	15.4%	3.8%	-15.6%	1578%	31.0%
			-2X six month horizon			
10%	15%	-20%	--	-20.1%	170%	0.1%
10%	30%	-20.2	0.2%	-27.8%	328%	7.6%
10%	45%	-24.9	6.2%	-39.1%	465%	14.1%
10%	60%	-37.2	21.6%	-52.1%	571%	14.8%
			-2X one year horizon			
10%	15%	-20%	--	-22.8%	337%	2.7%
10%	30%	-24.1	5.1%	-36.9%	627%	12.8%
10%	45%	-40.5	25.6%	-55.1%	841%	14.7%
10%	60%	-57.8	47.2%	-72.3%	965%	14.5%

Table 8: Selected results DR versus TB, 2X & –2X

of rebalancing based on price movements, rather than fixed time periods. The fixed-mix rules are ideally suited to trendless markets with considerable noise and low transaction costs. Conversely, momentum-like strategies perform best when markets exhibit trends with few reversals.

Levered and inverse funds are clearly derivatives of the underlying security. Thus, investors will naturally compare the performance of the levered fund to that of the underlying security. When the investor's horizon is greater than one day, term borrowed levered products are more consistent with most investors' expectations about the performance of these funds relative to that of the underlying asset.

The standard ETF approach, daily rebalancing, is a momentum-like strategy; it tends to outperform when underlying asset returns are trending in one direction or the other with relatively few reversals. In low volatility directional markets, daily rebalancing strategies are generally superior to buy-and-hold strategies because the amount levered (relative to the initial stake) increases as the underlying asset generally moves in one's favor with few, if any, reversals, thereby magnifying gains. Similarly, if the underlying asset generally moves in an adverse direction, leverage is steadily decreased, thereby mitigating losses and preventing the threat of early termination in all but the most extreme of circumstances.

Daily rebalancing performs poorly in trendless markets when there is a small or no change in the underlying asset value and in high volatility markets except for the most extreme movements. Over longer time periods, it is a tail strategy. Its relative performance generally worsens vis-à-vis the term borrowing strategy as the holding period increases. For the majority of outcomes, term borrowing performs better and increasingly so as volatility increases and as the holding period expands.

Lastly, daily rebalance ETFs buy and sell billions of dollars of market exposure over time, which increases turnover and transactions costs, generally resulting in reduced longer term investor returns vis-à-vis what is possible with term borrowing strategies. The macro-benefits of having a term borrowing based ETF solution versus the daily rebalance alternative may result in more stable investment environment and generally better overall return patterns for investors.

Rebalancing (or not rebalancing) decisions have a major impact on the performance of levered and inverse strategies. We have shown through empirical tests when daily rebalance leverage is likely to outperform term borrowing leverage, and vice versa. The performance characteristics of levered and inverse products should be well understood by investors before they invest in these products. Caveat emptor!

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